Anomalous Transport in Scale-Free Networks

Eduardo López,[1] Sergey V. Buldyrev,[2,1] Shlomo Havlin,[3,1] and H. Eugene Stanley[1]

[1]Center for Polymer Studies, Boston University, Boston, Massachusetts 02215, USA

[2]Department of Physics, Yeshiva University, 500 W. 185th Street, New York, New York 10033, USA

[3]Minerva Center & Department of Physics, Bar-Ilan University, Ramat Gan, Israel

(Received 30 November 2004; published 22 June 2005)

To study transport properties of scale-free and Erdős-Rényi networks, we analyze the conductance G between two arbitrarily chosen nodes of random scale-free networks with degree distribution $P(k) \sim k^{-\lambda}$ in which all links have unit resistance. We predict a broad range of values of G, with a power-law tail distribution $\Phi_{\rm SF}(G) \sim G^{-g_G}$, where $g_G = 2\lambda - 1$, and confirm our predictions by simulations. The power-law tail in $\Phi_{\rm SF}(G)$ leads to large values of G, signaling better transport in scale-free networks compared to Erdős-Rényi networks where the tail of the conductivity distribution decays exponentially. Based on a simple physical "transport backbone" picture we show that the conductances of scale-free and Erdős-Rényi networks are well approximated by $ck_Ak_B/(k_A + k_B)$ for any pair of nodes A and B with degrees k_A and k_B, where c emerges as the main parameter characterizing network transport.

DOI: 10.1103/PhysRevLett.94.248701 PACS numbers: 89.75.Hc, 05.60.Cd

Recent research on the topic of complex networks is leading to a better understanding of many real-world social, technological, and natural systems ranging from the World Wide Web and the Internet to cellular networks and sexual-partner networks [1]. A network topology that appears in many real-world systems is the scale-free network [2], characterized by a scale-free degree distribution: $P(k) \sim k^{-\lambda}$ and $k_{\min} \leq k \leq k_{\max}$, where k, the degree, is the number of links attached to a node. The cutoff value $k_{\min}$ represents the minimum allowed value of k on the network ($k_{\min} = 2$ here), and $k_{\max} \equiv k_{\min}N^{1/(\lambda-1)}$, the typical maximum degree of a network with N nodes [3,4]. The scale-free feature allows a network to have some nodes with a large number of links ("hubs"), unlike the case for the classic Erdős-Rényi model of random networks [5,6].

Here we characterize transport properties by conductance. We show that for scale-free networks with $\lambda \geq 2$, the conductance displays a power-law tail distribution that is related to the degree distribution $P(k)$. We find that the origin of the power-law tail is due to pairs of nodes of high degree. Thus, transport in scale-free networks is better than in Erdős-Rényi random networks. Also, we present a simple physical picture of transport in scale-free and Erdős-Rényi networks and test it with simulations.

The classic random networks of Erdős and Rényi [5,6] have a Poisson degree distribution, in contrast to the power-law distribution of the scale-free case. Because of the exponential decay of the degree distribution, the Erdős-Rényi networks (i) lack hubs and (ii) their properties, including transport, are controlled mainly by the average degree $\bar{k} \equiv \sum_{i=k_{\min}}^{k_{\max}} iP(i)$ [6,7].

Most of the work done so far regarding complex networks has concentrated on static topological properties or on models for their growth [1,3,8,9]. Transport features have not been extensively studied with the exception of random walks on complex networks [10–12], despite the fact that transport properties contain information about network function [13]. Here, we study the electrical conductance G between two nodes A and B of Erdős-Rényi and scale-free networks when a potential difference is imposed between them. We assume that all the links have equal resistances of unit value [14].

To construct an Erdős-Rényi network, we begin with N nodes and connect each pair with probability p. To generate a scale-free network with N nodes, we use the Molloy-Reed algorithm [15], which allows for the construction of random networks with arbitrary degree distribution. We generate k_i copies of each node i, where the probability of having k_i satisfies $P(k_i) \sim k_i^{-\lambda}$. These copies of the nodes are then randomly paired in order to construct the network, making sure that two previously linked nodes are not connected again, and also excluding links of a node to itself [16].

The conductance G of the network between two nodes A and B is calculated using the Kirchhoff method, where entering and exiting potentials are fixed to $V_A = 1$ and $V_B = 0$. We solve a set of linear equations to determine the potentials V_i of all nodes of the network. Finally, the total current $I \equiv G$ entering at node A and exiting at node B is computed by adding the outgoing currents from A to its nearest neighbors through $\Sigma_j(V_A - V_j)$, where j runs over the neighbors of A.

First, we analyze the probability density function (PDF) $\Phi(G)$ which comes from $\Phi(G)dG$, the probability that two nodes on the network have conductance between G and $G + dG$. To this end, we introduce the cumulative distribution $F(G) \equiv \int_G^\infty \Phi(G')dG'$, shown in Fig. 1(a) for the Erdős-Rényi and scale-free ($\lambda = 2.5$ and $\lambda = 3.3$, with $k_{\min} = 2$) cases. We use the notation $\Phi_{\rm SF}(G)$ and $F_{\rm SF}(G)$ for the scale-free case, and $\Phi_{\rm ER}(G)$ and $F_{\rm ER}(G)$ for the Erdős-Rényi case. The function $F_{\rm SF}(G)$ for both $\lambda = 2.5$

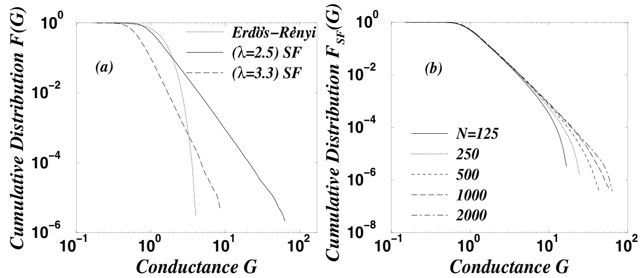

FIG. 1. (a) Comparison for networks with $N = 8000$ nodes between the cumulative distribution functions for the Erdős-Rényi and the scale-free cases (with $\lambda = 2.5$ and 3.3). Each curve represents the cumulative distribution $F(G)$ vs G. The simulations have at least 10^6 realizations. (b) Effect of system size on $F_{\mathrm{SF}}(G)$ vs G for the case $\lambda = 2.5$. The cutoff value of the maximum conductance $G_{\max}$ progressively increases as N increases.

and 3.3 exhibits a tail region well fit by the power law

$$F_{\mathrm{SF}}(G) \sim G^{-(g_G - 1)}, \tag{1}$$

and the exponent $(g_G - 1)$ increases with λ. In contrast, $F_{\mathrm{ER}}(G)$ decreases exponentially with G.

Increasing N does not significantly change $F_{\mathrm{SF}}(G)$ [Fig. 1(b)] except for an increase in the upper cutoff $G_{\max}$, where $G_{\max}$ is the typical maximum conductance, corresponding to the value of G at which $\Phi_{\mathrm{SF}}(G)$ crosses over from a power law to a faster decay. We observe no change of the exponent g_G with N. The increase of $G_{\max}$ with N implies that the average conductance $\bar{G}$ over all pairs also increases slightly [17].

We next study the origin of the large values of G in scale-free networks and obtain an analytical relation between λ and g_G. Larger values of G require the presence of many parallel paths, which we hypothesize arise from the high degree nodes. Thus, we expect that if either of the degrees k_A or k_B of the entering and exiting nodes is small, the conductance G between A and B is small since there are at most k different parallel branches coming out of a node with degree k. Thus, a small value of k implies a small number of possible parallel branches, and therefore a small value of G. To observe large G values, it is therefore necessary that both k_A and k_B be large.

We test this hypothesis by large scale computer simulations of the conditional PDF $\Phi_{\mathrm{SF}}(G|k_A, k_B)$ for specific values of the entering and exiting node degrees k_A and k_B. Consider first $k_B \ll k_A$, and the effect of increasing k_B, with k_A fixed. We find that $\Phi_{\mathrm{SF}}(G|k_A, k_B)$ is narrowly peaked [Fig. 2(a)] so that it is well characterized by G^*, the value of G when Φ_{SF} is a maximum. We find similar results for Erdős-Rényi networks. Further, for increasing k_B, we find [Fig. 2(b)] G^* increases as $G^* \sim k_B^{\alpha}$, with $\alpha = 0.96 \pm 0.05$ consistent with the possibility that as $N \to \infty$, $\alpha = 1$, which we assume henceforth.

For the case of $k_B \gtrsim k_A$, G^* increases less fast than k_B, as can be seen in Fig. 2(c) where we plot G^*/k_B against the scaled degree $x \equiv k_A/k_B$. The collapse of G^*/k_B for different values of k_A and k_B indicates that G^* scales as

$$G^* \sim k_B f\left(\frac{k_A}{k_B}\right). \tag{2}$$

The behavior of the scaling function $f(x)$ can be interpreted using the following simplified "transport backbone" picture [Fig. 2(c), inset], for which the effective conductance G between nodes A and B satisfies

$$\frac{1}{G} = \frac{1}{G_A} + \frac{1}{G_{\mathrm{tb}}} + \frac{1}{G_B}, \tag{3}$$

where $1/G_{\mathrm{tb}}$ is the resistance of the transport backbone while $1/G_A$ and $1/G_B$ are the resistances of the set of bonds near nodes A and B not belonging to the transport backbone. Based on Fig. 2(b), it is plausible that G_A is linear in k_A, so we can write $G_A = ck_A$. Since node B is equivalent to node A, we expect $G_B = ck_B$. Hence

$$\begin{aligned} G &= \frac{1}{1/ck_A + 1/ck_B + 1/G_{\mathrm{tb}}} \\ &= k_B \frac{ck_A/k_B}{1 + k_A/k_B + ck_A/G_{\mathrm{tb}}}, \end{aligned} \tag{4}$$

so the scaling function defined in Eq. (2) is

$$f(x) = \frac{cx}{1 + x + ck_A/G_{\mathrm{tb}}} \approx \frac{cx}{1 + x}. \tag{5}$$

The second equality follows if there are many parallel paths on the transport backbone so that $1/G_{\mathrm{tb}} \ll 1/ck_A$ [18]. The prediction (5) is plotted in Fig. 2(c) for both scale-free and Erdős-Rényi networks and the agreement with the simulations supports the approximate validity of the transport backbone picture of conductance in scale-free 1
and Erdős-Rényi networks.

The agreement of (5) with simulations has a striking implication: the conductance of a scale-free and Erdős-Rényi network depends on only one parameter c. Further, since the distribution of Fig. 2(a) is sharply peaked, a single measurement of G for any values of the degrees k_A and k_B of the entrance and exit nodes suffices to determine G^*, which then determines c and hence through Eq. (5) the conductance for all values of k_A and k_B.

Within this transport backbone picture, we can analytically calculate $F_{\mathrm{SF}}(G)$. Using Eq. (2), and the fact that $\Phi_{\mathrm{SF}}(G|k_A, k_B)$ is narrow, yields

$$\Phi_{\mathrm{SF}}(G) \sim \int P(k_B)dk_B \int P(k_A)dk_A \delta\left[k_B f\left(\frac{k_A}{k_B}\right) - G\right], \tag{6}$$

where $\delta(x)$ is the Dirac delta function. Performing the integration of Eq. (6) using (5), we obtain for $G < G_{\max}$

$$\Phi_{\mathrm{SF}}(G) \sim G^{-g_G} \qquad [g_G = 2\lambda - 1]. \tag{7}$$

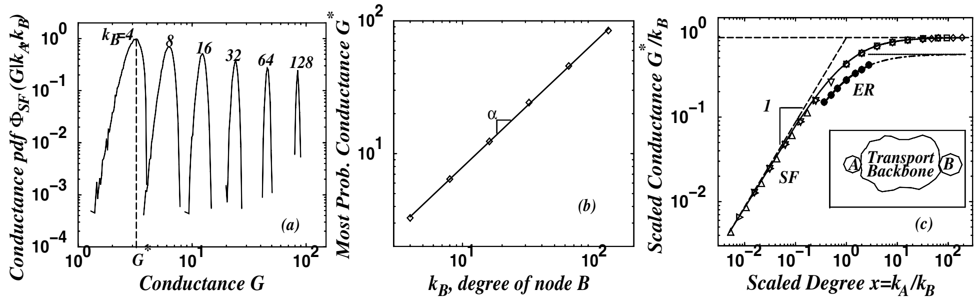

FIG. 2. (a) PDF $\Phi_{\rm SF}(G|k_A, k_B)$ vs G for $N = 8000$, $\lambda = 2.5$, and $k_A = 750$ (k_A is close to the typical maximum degree $k_{\max} = 800$ for $N = 8000$). (b) Most probable values G^*, estimated from the maxima of the distributions in Fig. 2(a), as a function of the degree k_B. The data support a power-law behavior $G^* \sim k_B^\alpha$ with $\alpha = 0.96 \pm 0.05$. (c) Scaled most probable conductance G^*/k_B vs scaled degree $x \equiv k_A/k_B$ for system size $N = 8000$ and $\lambda = 2.5$, for several values of k_A and k_B: $\square$ ($k_A = 8$, $8 \le k_B \le 750$), $\diamond$ ($k_A = 16$, $16 \le k_B \le 750$), $\triangle$ ($k_A = 750$, $4 \le k_B \le 128$), $\bigcirc$ ($k_B = 4$, $4 \le k_A \le 750$), ∇ ($k_B = 256$, $256 \le k_A \le 750$), and $\triangleright$ ($k_B = 500$, $4 \le k_A \le 128$). The curve crossing the symbols is the predicted function $G^*/k_B = f(x) = cx/(1+x)$ obtained from Eq. (5). We also show G^*/k_B vs scaled degree $x \equiv k_A/k_B$ for Erdős-Rényi networks with $\bar{k} = 2.92$, $4 \le k_A \le 11$, and $k_B = 4$ ($\bullet$), the curve crossing the symbols representing the theoretical result according to Eq. (5), and an extension of this line to represent the limiting value of G^*/k_B (dot-dashed line). The probability to obtain $k_A > 11$ is extremely small in Erdős-Rényi networks, and thus we are unable to obtain significant statistics. Scaling function $f(x)$, as seen here, exhibits a crossover from a linear behavior to the constant c ($c = 0.87 \pm 0.02$ for scale-free networks, horizontal dashed line, and $c = 0.55 \pm 0.01$ for Erdős-Rényi, dotted line). The inset shows a schematic of the "transport backbone" picture, where the circles labeled A and B denote nodes A and B and their associated links, which do not belong to the transport backbone.

Hence, for $F_{\rm SF}(G)$, we have $F_{\rm SF}(G) \sim G^{-(g_G-1)} \sim G^{-(2\lambda-2)}$. To test this prediction, we perform simulations for scale-free networks and calculate the values of $g_G - 1$ from the slope of a log-log plot of the cumulative distribution $F_{\rm SF}(G)$. From Fig. 3(b) we find that

$$g_G - 1 = (1.97 \pm 0.04)\lambda - (2.01 \pm 0.13). \qquad (8)$$

Thus, the measured slopes are consistent with the theoretical value predicted by Eq. (7) [19].

Next, we consider some further implications of our work. Our results show that larger values of G are found in scale-free networks with a much larger probability than in Erdős-Rényi networks, which raises the question whether scale-free networks have better transport than Erdős-Rényi networks. To answer this question, we consider the average conductance among all the pairs of nodes in the network, which quantifies how efficient the transport is. However, since scale-free networks are heterogeneous, we must find a way to assign proper weights to the choice of A and B. Recent work [20–22] suggests that the conductances of links between nodes i and j in certain real-world networks are characterized by $(k_i k_j)^\beta$, with $\beta = 1/2$. Using this weight, and comparing scale-free and Erdős-Rényi networks with equal values of $\bar{k}$ [23], we find that the average conductance of scale-free networks is larger than that of Erdős-Rényi networks (Table I). Even larger average conductance for scale-free networks compared to Erdős-Rényi networks (Table I) is obtained if one assumes [10] $\beta = 1$, i.e., that transport occurs with frequency proportional to the degree of the node. The case of $\beta = 0$ represents a "democratic" average, where all the

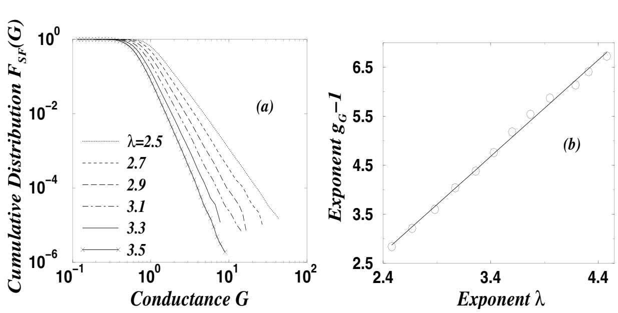

FIG. 3. (a) Simulation results for the cumulative distribution $F_{\rm SF}(G)$ for λ between 2.5 and 3.5, consistent with the power law $F_{\rm SF} \sim G^{-(g_G-1)}$ [cf. Eq. (7)], showing the progressive change of the slope $g_G - 1$. (b) The exponent $g_G - 1$ from simulations (circles) with $2.5 < \lambda < 4.5$; shown also is a least squares fit $g_G - 1 = (1.97 \pm 0.04)\lambda - (2.01 \pm 0.13)$, consistent with the predicted expression $g_G - 1 = 2\lambda - 2$ [cf. Eq. (7)].

TABLE I. Values of average conductance of scale-free and Erdős-Rényi networks for weights defined as $(k_i k_j)^\beta$. In parentheses we have indicated the values of the corresponding Erdős-Rényi networks.

Scale-free λ	$\bar{k}$	$\beta = 1$ $\bar{G}_{\rm SF}$ ($\bar{G}_{\rm ER}$)	$\beta = 1/2$ $\bar{G}_{\rm SF}$ ($\bar{G}_{\rm ER}$)	$\beta = 0$ $\bar{G}_{\rm SF}$ ($\bar{G}_{\rm ER}$)
2.5	5.3	5.5 (2.1)	2.4 (2.0)	1.3 (1.9)
2.7	4.3	2.7 (1.5)	1.8 (1.5)	1.1 (1.4)
2.9	3.7	1.7 (1.2)	1.4 (1.2)	0.9 (1.1)
3.1	3.4	1.3 (1.0)	1.1 (0.9)	0.8 (0.9)
3.3	3.1	1.0 (0.9)	1.0 (0.8)	0.7 (0.7)
3.5	2.9	0.8 (0.7)	0.8 (0.7)	0.6 (0.7)

pairs of nodes A and B are given the same weight. This case, which is not justified for heterogeneous networks, yields average conductance values for scale-free networks close to those of Erdős-Rényi networks (Table I). In many real-world systems, degree dependent link conductances and frequent use of high degree nodes both occur, making scale-free network transport even more efficient than Erdős-Rényi network transport.

In summary, we find a power-law tail for $\Phi_{SF}(G)$ and relate the tail exponent g_G to the exponent λ of the distribution $P(k)$. Our work is consistent with a simple physical picture of how transport takes place in scale-free and Erdős-Rényi networks.

We thank the Office of Naval Research, the Israel Science Foundation, and the European NEST project DYSONET for financial support, and L. Braunstein, S. Carmi, R. Cohen, E. Perlsman, G. Paul, S. Sreenivasan, T. Tanizawa, and Z. Wu for discussions.

Note added.—Three months after the submission of our manuscript to *Physical Review Letters* and its posting on xxx.lanl.gov, a similar manuscript appeared (Deok-Sun Lee and Heiko Rieger, cond-mat/0503008), which fully corroborates our findings.

[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002); R. Pastor-Satorras and A. Vespignani, *Structure and Evolution of the Internet: A Statistical Physics Approach* (Cambridge University Press, Cambridge, 2004); S. N. Dorogovtsev and J. F. F. Mendes, *Evolution of Networks: From Biological Nets to the Internet and WWW* (Oxford University Press, Oxford, 2003).

[2] A.-L. Barabási and R. Albert, Science **286**, 509 (1999); H. A. Simon, Biometrika **42**, 425 (1955).

[3] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. **85**, 4626 (2000).

[4] In principle, a node can have a degree up to $N - 1$, connecting to all other nodes of the network. The results presented here correspond to networks with upper cutoff $k_{\max} = k_{\min} N^{1/(\lambda-1)}$ imposed. We also studied networks for which $k_{\max}$ is not imposed, and found no significant differences in the PDF $\Phi_{SF}(G)$.

[5] P. Erdős and A. Rényi, Publ. Math. (Debrecen) **6**, 290 (1959).

[6] B. Bollobás, *Random Graphs* (Academic Press, Orlando, 1985).

[7] G. R. Grimmett and H. Kesten, J. Lond. Math. Soc. **30**, 171 (1984); math.PR/0107068.

[8] P. L. Krapivsky, S. Redner, and F. Leyvraz, Phys. Rev. Lett. **85**, 4629 (2000).

[9] Z. Toroczkai and K. Bassler, Nature (London) **428**, 716 (2004).

[10] J. D. Noh and H. Rieger, Phys. Rev. Lett. **92**, 118701 (2004).

[11] V. Sood, S. Redner, and D. ben-Avraham, J. Phys. A **38**, 109 (2005).

[12] L. K. Gallos, Phys. Rev. E **70**, 046116 (2004).

[13] The dynamical properties we study are related to transport on networks and differ from those that treat the network topology itself as evolving in time [2,8].

[14] The study of community structure has led some authors [M. E. J. Newman and M. Girvan, Phys. Rev. E **69**, 026113 (2004); F. Wu and B. A. Huberman, Eur. Phys. J. B **38**, 331 (2004)] to propose methods in which networks are considered as electrical networks in order to identify communities. In these studies, however, transport properties have not been addressed.

[15] M. Molloy and B. Reed, Random Struct. Algorithms **6**, 161 (1995).

[16] We performed simulations with the node copies randomly matched, and also matched in order of degree from highest to lowest, and obtained similar results.

[17] The average conductance $\bar{G}$ of networks depends on the weight given to different pairs of nodes [see discussion after Eq. (8) and Table I].

[18] Flux starts at node A, controlled by the conductance of the bonds in the vicinity of A. This flux passes into the "transport backbone," which is composed of many parallel paths and hence has a high conductance. Finally, flux ends at node B, controlled by the conductance of the bonds in the vicinity of B [see inset of Fig. 2(c)]. This is similar to traffic around a major freeway. Most of the limitations to transport occur in getting to the freeway ("node A") and then after leaving it ("node B"), but flow occurs easily on the freeway (transport backbone).

[19] The λ values explored here are limited by computer time considerations. In principle, we are unaware of any theoretical reason that would limit the validity of our results to any particular range of values of λ (above 2).

[20] E. Almaas, P. L. Krapivsky, and S. Redner, Phys. Rev. E **71**, 036124 (2005).

[21] K.-I. Goh, B. Kahng, and D. Kim, cond-mat/0410078.

[22] A. Barrat, M. Barthélemy, R. Pastor-Satorras, and A. Vespignani, Proc. Natl. Acad. Sci. U.S.A. **101**, 3747 (2004).

[23] The cost of a network is defined as its number of links. Two networks with equal N and $\bar{k}$ have equal cost. We compare each scale-free network with its corresponding equal cost Erdős-Rényi network.